ON THE COMPUTATION OF THE ECCENTRIC ANOMALY FROM THE MEAN ANOMALY OF A PLANET.

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THE relation between the eccentric and mean anomalies is given by the equation

$$m = E - e \sin E, \tag{1}$$

in which m is the mean anomaly, E the eccentric anomaly and e the eccentricity of the orbit.

In computing a series of values of E, it is not necessary to go beyond 180°, for any mean anomaly 360°—m corresponds to the eccentric anomaly 360°—E. Since E is greater than m, let us put

$$E = m + x,$$

where x is generally a small angle depending of course on the value of e.

In the case of the orbit of Mercury, whose eccentricity is .2056, the maximum value of x is less than 12°. Substituting in (1) we have

$$m = m + x - e \sin (m + x)$$

or

$$\begin{aligned} x &= e \sin (m + x) \\ &= e \sin m \cos x + e \cos m \sin x \\ &= e \sin m \left(1 - \frac{x^2}{1.2} + \ldots\right) + e \cos m \left(x - \frac{x^3}{1.2.3} + \ldots\right) \end{aligned}$$

Neglecting, for the first approximation, the third and all higher powers of x, we have

$$x^2 + \frac{2}{e}\left(\frac{1 - e \cos m}{\sin m}\right) x - 2 = 0, \tag{2}$$

whence

$$x = \sqrt{2} . \tan \tfrac{1}{2}\theta, \tag{3}$$

where

$$\tan \theta = e\sqrt{2} . \frac{\sin m}{1 - e \cos m}, \tag{4}$$

in which $\sqrt{2}$ is to be taken with the positive sign and θ always *less* than 90°.

The computation of the second member of the last equation will be much facilitated by the use of Zech's addition and subtraction Tables. Hence we have $E = m + x$ which will never differ more than $\pm 15''$ from the correct value of E, even when the eccentricity is as large as that of the orbit of Mercury. Let us represent this first approximate value of E, viz., $m + x$, by E_1, that is, let

$$E_1 = m + x. \tag{5}$$

Substituting E_1 for E in (1) we have

$$m_1 = E_1 - e \sin E_1, \tag{6}$$

which subtracted from (1) gives

$$\begin{aligned} m - m_1 &= E - E_1 - e(\sin E - \sin E_1) \\ &= E - E_1 - 2e \cos \tfrac{1}{2}(E + E_1) \sin \tfrac{1}{2}(E - E_1) \\ &= E - E_1 - e(E - E_1) \cos \tfrac{1}{2}(E + E_1) \\ &= (E - E_1)(1 - e \cos E_1) \text{ very nearly;} \end{aligned}$$

whence we have

$$E - E_1 = \frac{m - m_1}{1 - e \cos E_1}, \tag{7}$$

and $$E = E_1 + \frac{m - m_1}{1 - e \cos E_1},$$ for the second approxima-on. Therefore we have finally

$$E = m + x + \frac{m - m_1}{1 - e \cos E_1}. \tag{8}$$

If we require a third approximation, we may repeat the last operation with the corrected value of E as given by (8), but it will seldom or never be necessary as the last equation will generally give E within $0''.01$.

The second approximation given by (7) is substantially the same as that given by Gauss in his Theoria Motus, Art. 11, but the method here employed for obtaining it is much easier and more direct.

The second correction (7) is easily computed by the aid of Zech's Tables before referred to.

We will now test our formulæ by the following example:

Given $m = 143°$ and $e = .2056$, find E.

$\log \sqrt{2}$	$= 0.1505150$		
$\log e$	$= \bar{1}.3130231$		$= \bar{1}.3130231$
$\sin m$	$= 9.7794630$	$\cos m$	$= 9.9023486\,n$
Co. log $(1 - e \cos m)$	$= 9.9340098$ (Zech)		$9.2153717\,n$
$\tan \theta$	$= 9.1770109$	θ	$= 8°\ 32'\ 54''.8$
$\tan \frac{1}{2}\theta$	$= 8.8735468$		
$\sqrt{2}$	$= 0.1505150$		
cosec $1''$	$= 5.3144251$		
$\log x''$	$= 4.3384869$	x	$= 6°\ 3'21''.525$
		E_1	$= 149°\ 3'21''.525$
		$e \sin E_1$	$= 6°\ 3'26''.23$
		m_1	$= 142°59'55''.295$
		$m - m_1$	$= 4''.705.$

$$\log(m - m_1) = 0.6725596$$
$$\log(1 - e\cos E_1) = \underline{0.0705317} \text{ (Zech).}$$
$$\log 3''.9997 = 0.6020279.$$
$$\therefore E = m + x + 3''.9997$$
$$= 143^\circ + 6^\circ\ 3'21''.525 + 3''.9997$$
$$= 149^\circ\ 3'\ 25''.5247.$$

Check.
$$e\sin E = 6^\circ\ 3'\ 25''.52$$
$$\therefore m = E - e\sin E$$
$$= 143^\circ\ 0'\ 0''.0047.$$

If the second correction be taken at $4''$, which it is very nearly, E will be found to be $149^\circ 3' 25''.52$ exactly.

In computing a series of values of E, the labor may be lessened a little by preparing the constant logarithms, viz., $\log e\sqrt{2}$, $\log \sqrt{2} \operatorname{cosec} 1''$ and $\log e \operatorname{cosec} 1''$.

PLANE TRIGONOMETRY BY QUATERNIONS.

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A QUATERNION may be expressed under a variety of forms. Thus if a and b are the tensors respectively of the unit vectors α and β we have (see ANALYST, Vol. VII, pp. 124 and 127)

$$q = \frac{b}{a} i^t \tag{1}$$
$$= Tq.Uq \tag{2}$$
$$= Sq + Vq \tag{3}$$
$$= Tq(SUq + VUq). \tag{4}$$

Each of these forms has special advantages for the solution of certain problems, but in this article we will make use of the third one.

Two quaternions are equal when the elements of one equal respectively those of the other. Thus, if

$$q = q',$$

we have

$$Sq + Vq = Sq' + Vq', \tag{5}$$

and hence from the definition,

$$Sq = Sq', \text{ and } Vq = Vq'. \tag{6}$$

The principle in algebra corresponding to this is that where an equation is composed partly of real and partly of imaginary terms. Such an equation